SYZ w/ corrections II (cont'd)

Setting: (X, D) X : Kähler
D : effective anticanonical divisor with s.n.c.

assume $\exists$ a Lagr. torus fibration

$$\begin{array}{ccc} D & \subset & X \\ \downarrow & & \downarrow \pi \\ \partial B & \subset & B \end{array}$$

possibly w/ singular fibers over $B \setminus \partial B$

write $\Gamma \subset B \setminus \partial B$ the discriminant locus
$B_0 := (B \setminus \partial B) \setminus \Gamma$ the smooth locus

$\rightsquigarrow$

$$\begin{array}{ccc} T^*B_0/\Lambda^\vee \cong \pi^{-1}(B_0) =: X_0 & & \check{X}_0 := TB_0/\Lambda \\ \searrow & & \swarrow \\ & B_0 \subset B \setminus \partial B & \end{array}$$

- We need to correct/deform the complex structure $\check{J}_0$ on $\check{X}_0$, using instanton corrections from holom. disks in $X \setminus D$ with boundaries on regular Lagr. fibers of π, so that it extends to a suitable partial compactification of $\check{X}_0$. This should produce the correct mirror manifold $\check{X}$.
- D will give rise to a potential fcn $W : \check{X} \to \mathbb{C}$.

Auroux (2007): If $\exists$ sing. fibers in $\begin{array}{c} X \setminus D \\ \pi \downarrow \\ B \setminus \partial B \end{array}$ (i.e. $\Gamma \neq \emptyset$),

then before correcting the complex str. $\check{J}_0$,
the potential fcn W is multi-valued.
However, Floer theory tells us that W should be single-valued (or well-defined) on the corrected mirror

$\longrightarrow$ to find out the corrected complex str. on the mirror, we just need to see how W becomes a single-valued fcn.

§ Toric Calabi-Yau manifolds

Def A toric manifold $X = X_\Sigma$ is Calabi-Yau if $K_X \cong \mathcal{O}_X$.

This holds $\iff \exists\, u \in M$ s.t. $\langle u, v_i \rangle = 1 \ \forall i = 1, \dots, m$; here $v_1, \dots, v_m$ are primitive generators of rays in the fan Σ.

$$\begin{array}{ccc} v_1, & \dots, & v_m \\ \updownarrow & & \updownarrow \\ D_1, & \dots, & D_m \end{array} \quad \text{toric prime divisors}$$

$\iff \exists$ a holom. fcn $f : X \to \mathbb{C}$ (corr. to $u \in M$) s.t. $\operatorname{div}(f) = \sum_{i=1}^m D_i$

Rmk: Hence X is necessarily noncompact.

e.g. $X = \mathbb{C}^n$, $f = x_1 x_2 \cdots x_n$

$$\Omega = dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

Now suppose $T^n \curvearrowright X$ is an n-dim$^{\text{nal}}$ toric CY mfd. Consider the $T^{n-1} \subset T^n$ which preserves the holom. volume form Ω. (or f)

Prop (Gross, Goldstein)

The map

$$\pi : X \longrightarrow \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} =: B$$
$$p \longmapsto \left(\mu_{T^{n-1}}(p), |f(p) - \varepsilon|\right) \quad \left(\text{for some fixed } \varepsilon \in \mathbb{R}_{>0}\right)$$

is a Lagrangian torus fibration with $\pi^{-1}(\partial B) = \{ p \in X \mid f(p) = \varepsilon \} =: D_\varepsilon$.

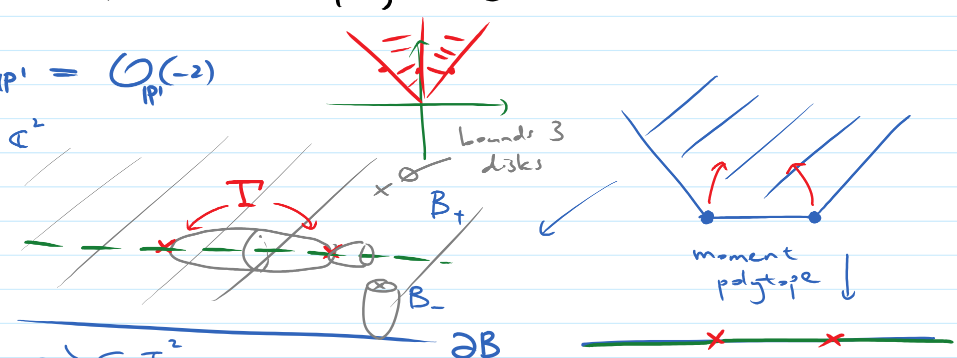
The discriminant locus is an explicit real codim 2 subset Γ contained in the hyperplane (the wall)

$$H := \mathbb{R}^{n-1} \times \{\varepsilon\} \subset B$$

e.g. ① $X \overset{T^2}{\curvearrowleft} = K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$

$\| \; \mathbb{C}^2 \cup \mathbb{C}^2$

$B =$



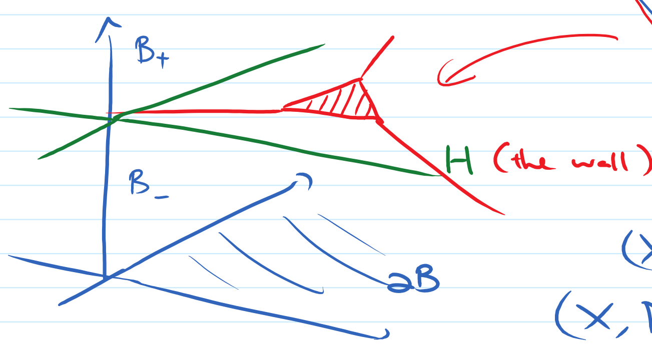
$T^2 \curvearrowright K_{\mathbb{P}^1}$

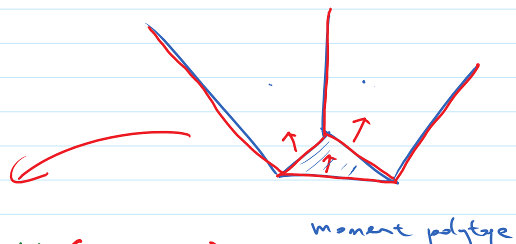
$(e^{i\theta_1}x, e^{i\theta_2}y)$

$S^1 = \{\theta_1 + \theta_2 = 0\} \subset T^2$

② $X = K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$

$B =$





$(X, D_\varepsilon) \quad \varepsilon \to 0^+$

$(X, D_0) \hookrightarrow ((\mathbb{C}^*)^n, W)$

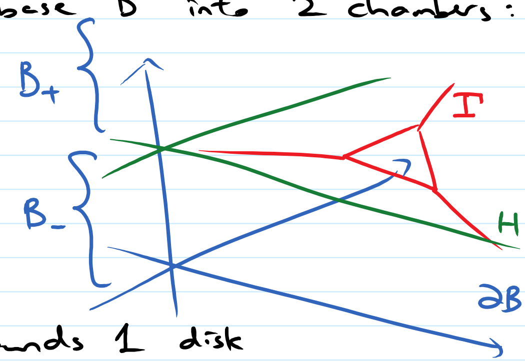
$W = W_{Floer} = \sum_{i=1}^{m} (1+\delta_i(q)) z^{\beta_i}$

Now we want to use the fibration $\pi : X \to B$ to study the mirror symmetry for (X, D_ε).

The wall H divides the base B into 2 chambers:

$$B_+ = \mathbb{R}^{n-1} \times (\varepsilon, +\infty)$$

$$B_- = \mathbb{R}^{n-1} \times (0, \varepsilon)$$



If we compute W, then fiber over a pt in B_- only bounds 1 disk

$$\longrightarrow \quad W_- = u$$

however, fiber over a pt in B_- bounds a lot more disks

$$\longrightarrow \quad W_+ = \sum_{i=1}^{m} (1 + \delta_i(q))\, y \cdot Z_{\beta_i}$$

We can conclude that the SYZ mirror of a toric CY of X is given by

$-S_{\beta_i}^{\omega}$

We can conclude that the SYZ mirror of a toric CY mfd X is given by

$$\check{X}_{SYZ} := \left\{ (u,v,\vec{z}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} \;\middle|\; uv = \sum_{i=1}^{m} (1+\delta_i(q))\, z_{\beta_i} \right\}$$

$e^{-\int_{\beta_i}\omega}\, \mathrm{hol}_\nabla(\partial\beta_i)$

where
$$1+\delta_i(q) = \sum_{\substack{\alpha \in H_2(X;\mathbb{Z}) \\ c_1(\alpha)=0}} n_{\beta_i+\alpha} \cdot q^\alpha, \qquad n_{\beta_i+\alpha} = (ev_0)_*\left([\overline{\mathcal{M}}_1(L,\beta_i+\alpha)]^{vir}\right) \in \hat{\mathbb{Q}}$$

rep. by a configuration w/ sphere bubbles

$q^\alpha = e^{-\int_\alpha \omega}$

$\rightsquigarrow (X, D_\varepsilon)$ is mirror to $(\check{X}_{SYZ}, W)$, $W = u$

e.g. for $X = K_{\mathbb{P}^1}$,

$$\check{X}_{SYZ} = \left\{ uv = 1 + q + z + \frac{q}{z} \right\} \subset \mathbb{C}^2_{u,v} \times \mathbb{C}^\times_z$$

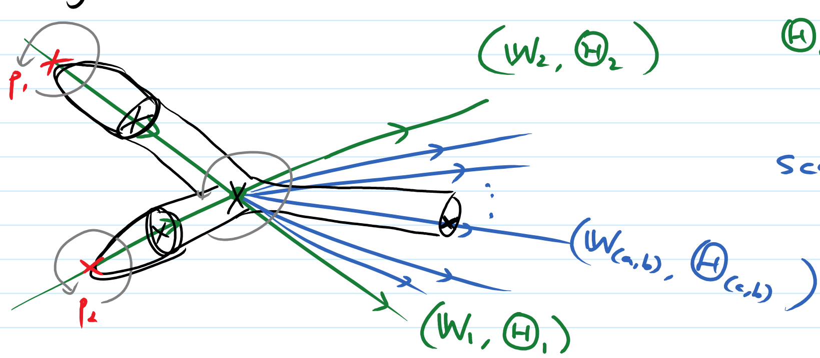
for $X = K_{\mathbb{P}^2}$,

$$\check{X}_{SYZ} = \left\{ uv = 1+\delta_0(q) + z_1 + z_2 + \frac{q}{z_1 z_2} \right\}$$

where $1+\delta_0(q) = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \cdots$

How about more general cases?

In general, there are more than one walls



Θ_1, Θ_2: wall-crossing factors

scattering diagram

holom disk/curve on X

cpx str. on $\check{X}$

Floer, Fukaya-Oh

Kontsevich-Soibelman, Fukaya, Gross-Siebert

scattering diagrams